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Twisted Supersymmetry and Non-Anticommutative Superspace

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Abstract: We extend the analysis of hep-th/0408069 on a Lorentz invariant interpretation of noncommutative spacetime to field theories on non-anticommutative superspace with half the supersymmetries broken. By defining a Drinfeld-twisted Hopf superalgebra, it is shown that one can restore *twisted* supersymmetry and therefore obtain a twisted version of the chiral rings along with certain Ward-Takahashi identities. Moreover, we argue that the representation content of theories on the deformed superspace is identical to that of their undeformed cousins and comment on the consequences of our analysis concerning non-renormalization theorems.

Keywords: Quantum Groups, Extended Supersymmetry, Non-Commutative Geometry, Superspaces.

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Contents

1. Introduction

Over the last decade, there has been an immense effort by string theorists to improve our understanding of string dynamics in nontrivial backgrounds. Most prominently, Seiberg and Witten [1] discovered that superstring theory in a constant Kalb-Ramond 2-form background can be formulated in terms of field theories on noncommutative spacetimes upon taking the so-called Seiberg-Witten zero slope limit. Subsequently, these noncommutative variants of ordinary field theories were intensely studied. It turned out that as low energy effective field theories, noncommutative field theories exhibit many manifestations of stringy features descending from the underlying string theory. Therefore, these theories have proven to be an ideal toy model for studying string theoretic questions which otherwise remain intractable.

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Recently, it was realized that noncommutative field theories, although manifestly breaking Poincaré symmetry,[1] can be recast into a form which is invariant under a twist-deformed action of the Poincaré algebra [2 – 4]. In this framework, the commutation relation $[x^\mu, x^\nu] = \mathrm{i}\Theta^{\mu\nu}$ is understood as a result of the non-*co*commutativity of the coproduct of a twisted Hopf Poincaré algebra acting on the coordinates. This result can be used to show that the representation content of Moyal-Weyl-deformed theories is identical to that of their undeformed Lorentz invariant counterparts. Furthermore, theorems in quantum field theory which require Lorentz invariance for their proof can now be carried over to the Moyal-Weyl-deformed case using twisted Lorentz invariance. For related works, see [5 – 10].

The purpose of the present paper is to extend the analysis of [3, 4] to supersymmetric field theories on non-anticommutative superspaces. The latter spaces naturally arise when considering type-II superstring theories in a constant graviphoton background. In [11], Seiberg showed that there is a deformation of euclidean $\mathcal{N} = 1$ superspace[2] in four dimensions which leads to a consistent supersymmetric field theory with half of the supersymmetries broken. Using the Hopf-algebra twist, one can render twisted supersymmetry manifest and use it to preserve properties of supersymmetric field theories in the non-anticommutative situation. Note that twisted supersymmetry was already considered in [14]. However, the analysis of extended supersymmetries presented in this reference differs from the one we will propose here. Furthermore, we will discuss several new applications of the re-gained twisted supersymmetry.

The paper is organized as follows: We will fix our conventions for non-anticommutative superspace in section 2 and introduce the Drinfeld twist of the euclidean super Poincaré algebra and its universal enveloping algebra in section 3. Then, in section 4, we will discuss the implications of these mathematical structures concerning the representation content, the reemergence of (twisted) chiral rings and Ward-Takahashi identities. Moreover, we comment on non-renormalization theorems in the twisted supersymmetric case before we conclude in section 5. Some basic definitions and a useful extension of the Baker-Campbell-Hausdorff formula can be found in the appendix.

2. Non-anticommutative superspace

2.1 Superspace conventions

Throughout this paper, we will mostly adopt the conventions of [11]. Consider the four-dimensional euclidean space $\mathbb{R}^4$ with coordinates (x^μ) and extend it to the space $\mathbb{R}^{4|4\mathcal{N}}$ by adding $4\mathcal{N}$ Graßmann coordinates $(\theta^{i\alpha}, \bar\theta_i^{\dot\alpha})$ with[3] $i = 1, \ldots, \mathcal{N}$ and $\alpha, \dot\alpha = 1, 2$. The

[1] In noncommutative spacetime, the Poincaré group is broken down to the stabilizer subgroup of the deformation tensor.

[2] Later on, non-anticommutativity for extended supersymmetry was considered, as well [12, 13].

[3] Strictly speaking, this superspace with euclidean signature can be consistently defined only for an even number of supersymmetries, as the appropriate reality condition for $\theta^{i\alpha}$ and $\bar\theta_i^{\dot\alpha}$ is a symplectic Majorana condition and establishes a pairwise relation between these spinors. When working on the complexified superspace $\mathbb{C}^{4|4\mathcal{N}}$, i.e., when "temporarily doubling the fermionic degrees of freedom", this obstacle however disappears.

algebra of superfunctions on this space is denoted by

$$\mathcal{S} \; := \; C^\infty \otimes \Lambda_{4\mathcal{N}} \; , \tag{2.1}$$

where $\Lambda_{4\mathcal{N}} := \Lambda^\bullet(\mathbb{R}^{4\mathcal{N}})$ is the Graßmann algebra with $4\mathcal{N}$ generators. As it is well known, an element of $\mathcal{S}$ can be decomposed into its Graßmann even and its Graßmann odd part as well as into its "body" (the purely bosonic part) and its "soul" (the nilpotent part), cf. e.g. [15].

Recall that translations in the Graßmann directions of this space are generated by the superderivatives and the supercharges which act on a superfunction $f \in \mathcal{S}$ as

$$\begin{aligned} D_{i\alpha} f \; &:= \; \frac{\partial}{\partial\theta^{i\alpha}} f + \mathrm{i}\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}_i \partial_\mu f \; , \quad \bar D^i_{\dot\alpha} f \; := \; -\frac{\partial}{\partial\bar\theta^{\dot\alpha}_i} f - \mathrm{i}\theta^{i\alpha}\sigma^\mu_{\alpha\dot\alpha}\partial_\mu f \; , \\ Q_{i\alpha} f \; &:= \; \frac{\partial}{\partial\theta^{i\alpha}} f - \mathrm{i}\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}_i \partial_\mu f \; , \quad \bar Q^i_{\dot\alpha} f \; := \; -\frac{\partial}{\partial\bar\theta^{\dot\alpha}_i} f + \mathrm{i}\theta^{i\alpha}\sigma^\mu_{\alpha\dot\alpha}\partial_\mu f \; , \end{aligned} \tag{2.2}$$

and satisfy the algebra (we do not allow for central charges)

$$\begin{aligned} &\{D_{i\alpha}, D_{j\beta}\} = 0 \; , \quad \{\bar D^i_{\dot\alpha}, \bar D^j_{\dot\beta}\} = 0 \; , \quad \{D_{i\alpha}, \bar D^j_{\dot\beta}\} = -2\mathrm{i}\delta^j_i \sigma^\mu_{\alpha\dot\beta}\partial_\mu \; , \\ &\{Q_{i\alpha}, Q_{j\beta}\} = 0 \; , \quad \{\bar Q^i_{\dot\alpha}, \bar Q^j_{\dot\beta}\} = 0 \; , \quad \{Q_{i\alpha}, \bar Q^j_{\dot\beta}\} = 2\mathrm{i}\delta^j_i \sigma^\mu_{\alpha\dot\beta}\partial_\mu \; . \end{aligned} \tag{2.3}$$

Our discussion simplifies considerably if we switch to the left-handed chiral coordinates

$$(y^\mu := x^\mu + \mathrm{i}\theta^{i\alpha}\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}_i , \; \theta^{i\alpha} , \; \bar\theta^{\dot\alpha}_i) \; , \tag{2.4}$$

in which the representations of the superderivatives and the supercharges read

$$\begin{aligned} D_{i\alpha} f &= \frac{\partial}{\partial\theta^{i\alpha}} f + 2\mathrm{i}\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}_i \partial^L_\mu f \; , \quad \bar D^i_{\dot\alpha} f = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}_i} f \; , \\ Q_{i\alpha} f &= \frac{\partial}{\partial\theta^{i\alpha}} f \; , \qquad\qquad\qquad\quad \bar Q^i_{\dot\alpha} f = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}_i} f + 2\mathrm{i}\theta^{i\alpha}\sigma^\mu_{\alpha\dot\alpha}\partial^L_\mu f \; , \end{aligned} \tag{2.5}$$

where ∂^L_μ denotes a derivative with respect to y^μ. Due to $\partial^L_\mu = \partial_\mu$, we drop the superscript "L" in the following.

2.2 The canonical non-anticommutative deformation

The canonical deformation of $\mathbb{R}^{4|4\mathcal{N}}$ to $\mathbb{R}^{4|4\mathcal{N}}_\hbar$ amounts to [11]

$$\{\hat\theta^{i\alpha}, \hat\theta^{j\beta}\} = \hbar C^{i\alpha,j\beta} \; , \tag{2.6}$$

where the hats indicate, as usual, that we are dealing with an operator representation. Equivalently, one can instead deform the algebra of superfunctions $\mathcal{S}$ on $\mathbb{R}^{4|4\mathcal{N}}$ to an algebra $\mathcal{S}_\star$, in which the product is given by the Moyal-type star product

$$f \star g = f \exp\left(-\frac{\hbar}{2} \overleftarrow{Q}_{i\alpha} C^{i\alpha,j\beta} \overrightarrow{Q}_{j\beta} \right) g \; , \tag{2.7}$$

where $\overleftarrow{Q}_{i\alpha}$ and $\overrightarrow{Q}_{j\beta}$ are supercharges acting from the right and the left, respectively. Recall that $\theta^{i\alpha}\overleftarrow{Q}_{j\beta} = -\delta^i_j\delta^\alpha_\beta$. All commutators involving this star multiplication will be denoted by a $\star$, e.g. the graded commutator will read

$$\{\!\![f,g]\!\!\}_\star \ := \ f \star g - (-1)^{\tilde{f}\tilde{g}} g \star f \ . \tag{2.8}$$

with $\tilde{f}$ and $\tilde{g}$ denoting the grading of f and g, respectively, cf. appendix A.

In chiral coordinates, we have the following coordinate algebra on $\mathcal{S}_\star$:

$$\begin{aligned} \{\theta^{i\alpha},\theta^{j\beta}\}_\star &= \hbar C^{i\alpha,j\beta} \ , \\ [y^\mu,y^\nu]_\star = [y^\mu,\theta^{i\alpha}]_\star = [y^\mu,\theta^{i\alpha}]_\star \ &= \ \{\theta^{i\alpha},\bar{\theta}^{\dot\alpha}_i\}_\star = \{\bar{\theta}^{\dot\alpha}_i,\bar{\theta}^{\dot\beta}_j\}_\star = 0 \ . \end{aligned} \tag{2.9}$$

This deformation has been shown to arise in string theory from open superstrings of type-IIB in the background of a constant graviphoton field strength [16, 11, 17]. The corresponding deformed algebra of superderivatives and supercharges reads

$$\begin{aligned} &\{D_{i\alpha},D_{j\beta}\}_\star = 0 \ , \qquad \{\bar{D}^i_{\dot\alpha},\bar{D}^j_{\dot\beta}\}_\star = 0 \ , \\ &\{D_{i\alpha},\bar{D}^j_{\dot\beta}\}_\star = -2\mathrm{i}\delta^j_i\sigma^\mu_{\alpha\dot\beta}\partial_\mu \ , \\ &\{Q_{i\alpha},Q_{j\beta}\}_\star = 0 \ , \qquad \{\bar{Q}^i_{\dot\alpha},\bar{Q}^j_{\dot\beta}\}_\star = -4\hbar C^{i\alpha,j\beta}\sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta}\partial_\mu\partial_\nu \ , \\ &\{Q_{i\alpha},\bar{Q}^j_{\dot\beta}\}_\star = 2\mathrm{i}\delta^j_i\sigma^\mu_{\alpha\dot\beta}\partial_\mu \ . \end{aligned} \tag{2.10}$$

By inspection of this deformed algebra, it becomes clear that the number of supersymmetries is reduced to $\mathcal{N}/2$, since those generated by $\bar{Q}^i_{\dot\alpha}$ will be broken.[4] On the other hand, it still allows for the definition of chiral and anti-chiral superfields as the algebra of the superderivatives $D_{i\alpha}$ and $\bar{D}^i_{\dot\alpha}$ is undeformed.

An alternative approach, which was followed in [18], manifestly preserves supersymmetry but breaks chirality. This simply means that one replaces the supercharges $Q_{i\alpha}$ by the superderivatives $D_{i\alpha}$ in the definition of the deformation (2.7). Without chiral superfields, however, it is impossible to define super Yang-Mills theory in the standard superspace formalism.

In the approach we will present in the following, supersymmetry *and* chirality are manifestly and simultaneously preserved, albeit in a twisted form.

3. Drinfeld twist of the euclidean super Poincaré algebra

3.1 The euclidean super Poincaré algebra and its enveloping algebra

The starting point of our discussion is the ordinary euclidean super Poincaré algebra[5] $\mathfrak{g}$ on $\mathbb{R}^{4|4\mathcal{N}}$ without central extensions, which generates the isometries on the space $\mathbb{R}^{4|4\mathcal{N}}$. More

[4] Note that this result is due to the fact that we are using euclidean spacetime. In Minkowski superspace, Q and $\bar{Q}$ are related via complex conjugation, and it is therefore not possible to break only half of the supersymmetries.

[5] or inhomogeneous super euclidean algebra

explicitly, we have the generators of translations P_μ, the generators of four-dimensional rotations $M_{\mu\nu}$ and the $4\mathcal{N}$ supersymmetry generators $Q_{i\alpha}$ and $\bar{Q}^i_{\dot\alpha}$. They satisfy the following algebra:

$$\begin{aligned}
[P_\rho, M_{\mu\nu}] &= \mathrm{i}(\delta_{\mu\rho} P_\nu - \delta_{\nu\rho} P_\mu)\ , \\
[M_{\mu\nu}, M_{\rho\sigma}] &= -\mathrm{i}(\delta_{\mu\rho} M_{\nu\sigma} - \delta_{\mu\sigma} M_{\nu\rho} - \delta_{\nu\rho} M_{\mu\sigma} + \delta_{\nu\sigma} M_{\mu\rho})\ , \\
[P_\mu, Q_{i\alpha}] &= 0\ , \qquad\qquad\qquad [P_\mu, \bar{Q}^i_{\dot\alpha}] = 0\ , \\
[M_{\mu\nu}, Q_{i\alpha}] &= \mathrm{i}(\sigma_{\mu\nu})_\alpha{}^\beta Q_{i\beta}\ , \quad [M_{\mu\nu}, \bar{Q}^{i\dot\alpha}] = \mathrm{i}(\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta} \bar{Q}^{i\dot\beta}\ , \\
\{Q_{i\alpha}, \bar{Q}^j_{\dot\beta}\} &= 2\delta_i^j \sigma^\mu_{\alpha\dot\beta} P_\mu\ , \qquad \{Q_{i\alpha}, Q_{j\beta}\} = \{\bar{Q}^i_{\dot\alpha}, \bar{Q}^j_{\dot\beta}\} = 0\ .
\end{aligned} \tag{3.1}$$

The Casimir operators of the Poincaré algebra used for labelling representations are P^2 and W^2, where the latter is the square of the Pauli-Ljubanski operator

$$W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma} M^{\nu\rho} P^\sigma\ . \tag{3.2}$$

This operator is, however, not a Casimir of the super Poincaré algebra; instead, there is a supersymmetric variant: the (superspin) operator $\widetilde{C}^2$ defined as the square of

$$\widetilde{C}_{\mu\nu} = \widetilde{W}_\mu P_\nu - \widetilde{W}_\nu P_\mu, \tag{3.3}$$

where $\widetilde{W}_\mu := W_\mu - \frac{1}{4}\bar{Q}^i_{\dot\alpha}\bar\sigma_\mu^{\dot\alpha\alpha} Q_{i\alpha}$.

Recall that a *universal enveloping algebra* $\mathcal{U}(\mathfrak{a})$ of a Lie algebra $\mathfrak{a}$ is an associative unital algebra together with a Lie algebra homomorphism $h : \mathfrak{a} \to \mathcal{U}(\mathfrak{a})$, satisfying the following universality property: For any further associative algebra A with homomorphism $\phi : \mathfrak{a} \to A$, there exists a unique homomorphism $\psi : \mathcal{U}(\mathfrak{a}) \to A$ of associative algebras, such that $\phi = \psi \circ h$. Every Lie algebra has an universal enveloping algebra, which is unique up to algebra isomorphisms.

The univeral enveloping algebra $\mathcal{U}(\mathfrak{g})$ of the euclidean super Poincaré algebra $\mathfrak{g}$ is a cosupercommutative Hopf superalgebra[6] with counit and coproduct defined by $\varepsilon(\mathbb{1}) = 1$ and $\varepsilon(x) = 0$ otherwise, $\Delta(\mathbb{1}) = \mathbb{1} \otimes \mathbb{1}$ and $\Delta(x) = \mathbb{1} \otimes x + x \otimes \mathbb{1}$ otherwise.

3.2 The Drinfeld twist of the enveloping algebra

Given a Hopf algebra H with coproduct Δ, a counital 2-cocycle $\mathcal{F}$ is a counital element of $H \otimes H$, which has an inverse and satisfies

$$\mathcal{F}_{12}(\Delta \otimes \mathrm{id})\mathcal{F} = \mathcal{F}_{23}(\mathrm{id} \otimes \Delta)\mathcal{F}\ , \tag{3.4}$$

where we used the common shorthand notation $\mathcal{F}_{12} = \mathcal{F} \otimes \mathbb{1}$, $\mathcal{F}_{23} = \mathbb{1} \otimes \mathcal{F}$ etc. As done in [3], such a counital 2-cocycle $\mathcal{F} \in H \otimes H$ can be used to define a twisted Hopf algebra[7] $H^{\mathcal{F}}$ with a new coproduct given by

$$\Delta^{\mathcal{F}}(Y)\ :=\ \mathcal{F}\Delta(Y)\mathcal{F}^{-1}\ . \tag{3.5}$$

[6] cf. appendix

[7] This twisting amounts to constructing a quasitriangular Hopf algebra, as discussed, e.g., in [19].

The element $\mathcal{F}$ is called a *Drinfeld twist*; such a construction was first considered in [20].

For our purposes, i.e. to recover the canonical algebra of non-anticommutative coordinates (2.9), we choose the abelian twist $\mathcal{F} \in \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ defined by

$$\mathcal{F} = \exp\left(-\frac{\hbar}{2} C^{i\alpha,j\beta} Q_{i\alpha} \otimes Q_{j\beta}\right) . \tag{3.6}$$

As one easily checks, $\mathcal{F}$ is indeed a counital 2-cocycle: First, it is invertible and its inverse is given by $\mathcal{F}^{-1} = \exp\left(\frac{\hbar}{2} C^{i\alpha,j\beta} Q_{i\alpha} \otimes Q_{j\beta}\right)$. (Because the $Q_{i\alpha}$ are nilpotent, $\mathcal{F}$ and $\mathcal{F}^{-1}$ are not formal series but rather finite sums.) Second, $\mathcal{F}$ is counital since it satisfies the conditions

$$(\varepsilon \otimes \mathrm{id})\mathcal{F} = \mathbb{1} \qquad \text{and} \qquad (\mathrm{id} \otimes \varepsilon)\mathcal{F} = \mathbb{1} , \tag{3.7}$$

as can be verified without difficulty. Also, the remaining cocycle condition (3.4) turns out to be fulfilled since

$$\begin{aligned} \mathcal{F}_{12}(\Delta \otimes \mathrm{id})\mathcal{F} &= \mathcal{F}_{12} \exp\left(-\frac{\hbar}{2} C^{i\alpha,j\beta} (Q_{i\alpha} \otimes \mathbb{1} + \mathbb{1} \otimes Q_{i\alpha}) \otimes Q_{j\beta}\right) , \\ \mathcal{F}_{23}(\mathrm{id} \otimes \Delta)\mathcal{F} &= \mathcal{F}_{23} \exp\left(-\frac{\hbar}{2} C^{i\alpha,j\beta} Q_{i\alpha} \otimes (Q_{j\beta} \otimes \mathbb{1} + \mathbb{1} \otimes Q_{j\beta})\right) \end{aligned} \tag{3.8}$$

yields, due to the commutativity of the $Q_{i\alpha}$,

$$\mathcal{F}_{12}\mathcal{F}_{13}\mathcal{F}_{23} = \mathcal{F}_{23}\mathcal{F}_{12}\mathcal{F}_{13} , \tag{3.9}$$

which is obviously true.

Note that after introducing this Drinfeld twist, the multiplication in $\mathcal{U}(\mathfrak{g})$ and the action of $\mathfrak{g}$ on the coordinates remain the same. In particular, the representations of the twisted and the untwisted algebras are identical. It is only the action of $\mathcal{U}(\mathfrak{g})$ on the tensor product of the representation space, given by the coproduct, which changes.

Let us be more explicit on this point: the coproduct of the generator P_μ does not get deformed, as P_μ commutes with $Q_{j\beta}$:

$$\Delta^{\mathcal{F}}(P_\mu) = \Delta(P_\mu) . \tag{3.10}$$

For the other generators of the euclidean super Poincaré algebra, the situation is slightly more complicated. Due to the rule[8] $(a_1 \otimes a_2)(b_1 \otimes b_2) = (-1)^{\tilde{a}_2\tilde{b}_1}(a_1 b_1 \otimes a_2 b_2)$, where $\tilde{a}$ denotes the Graßmann parity of a, we have the relations[9] (cf. equation (B.2))

$$\begin{aligned} &\mathcal{F}\,(D \otimes \mathbb{1})\,\mathcal{F}^{-1} = \\ &\quad = \sum_{n=0}^{\infty} \frac{(-1)^{n\tilde{D} + \frac{n(n-1)}{2}}}{n!} \left(-\frac{\hbar}{2}\right)^n C^{I_1 J_1} \dots C^{I_n J_n} [\![Q_{I_1}, [\![\dots [\![Q_{I_n}, D]\!]]\!]]\!] \otimes Q_{J_1} \dots Q_{J_n} , \end{aligned}$$

[8] cf. appendix

[9] Here, I_k and J_k are multi-indices, e.g. $I_k = i_k \alpha_k$.

$$\mathcal{F}\,(\mathbb{1}\otimes D)\,\mathcal{F}^{-1} = \qquad (3.11)$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^{n\tilde{D}+\frac{n(n-1)}{2}}}{n!}\left(-\frac{\hbar}{2}\right)^{n} C^{I_1J_1}\dots C^{I_nJ_n} Q_{I_1}\dots Q_{I_n}\otimes [\![Q_{J_1},[\![\dots[\![Q_{J_n},D]\!]]\!]]\!] \ ,$$

where $[\![\cdot,\cdot]\!]$ denotes the graded commutator. From this, we immediately obtain

$$\Delta^{\mathcal{F}}(Q_{i\alpha}) = \Delta(Q_{i\alpha}) \ . \qquad (3.12)$$

Furthermore, we can also derive the expressions for $\Delta^{\mathcal{F}}(M_{\mu\nu})$ and $\Delta^{\mathcal{F}}(\bar{Q}^k_{\dot{\gamma}})$, which read

$$\Delta^{\mathcal{F}}(M_{\mu\nu}) = \Delta(M_{\mu\nu}) + \frac{\mathrm{i}\hbar}{2}C^{i\alpha,j\beta}\left[(\sigma_{\mu\nu})_\alpha{}^\gamma Q_{i\gamma}\otimes Q_{j\beta} + Q_{i\alpha}\otimes(\sigma_{\mu\nu})_\beta{}^\gamma Q_{j\gamma}\right] \ , \qquad (3.13)$$
$$\Delta^{\mathcal{F}}(\bar{Q}^k_{\dot{\gamma}}) = \Delta(\bar{Q}^k_{\dot{\gamma}}) + \hbar C^{i\alpha,j\beta}\left[\delta^k_i\sigma^\mu_{\alpha\dot{\gamma}}P_\mu\otimes Q_{j\beta} + Q_{i\alpha}\otimes\delta^k_j\sigma^\mu_{\beta\dot{\gamma}}P_\mu\right] . \qquad (3.14)$$

The twisted coproduct of the Pauli-Ljubanski operator W_μ becomes

$$\Delta^{\mathcal{F}}(W_\mu) = \Delta(W_\mu) - \frac{\mathrm{i}\hbar}{4}C^{i\alpha,j\beta}\epsilon_{\mu\nu\rho\sigma}\left(Q_{i\alpha}\otimes(\sigma^{\nu\rho})_\beta{}^\gamma Q_{j\gamma}P^\sigma + (\sigma^{\nu\rho})_\alpha{}^\gamma Q_{i\gamma}P^\sigma\otimes Q_{j\beta}\right) \ , \quad (3.15)$$

while for its supersymmetric variant $\widetilde{C}_{\mu\nu}$, we have

$$\begin{aligned} \Delta^{\mathcal{F}}(\widetilde{C}_{\mu\nu}) &= \Delta(\widetilde{C}_{\mu\nu}) - \frac{\hbar}{2}C^{i\alpha,j\beta}\left[Q_{i\alpha}\otimes Q_{j\beta},\Delta(\widetilde{C}_{\mu\nu})\right] \\ &= \Delta(\widetilde{C}_{\mu\nu}) - \frac{\hbar}{2}C^{i\alpha,j\beta}\left(\left[Q_{i\alpha},\widetilde{C}_{\mu\nu}\right]\otimes Q_{j\beta} + Q_{i\alpha}\otimes\left[Q_{j\beta},\widetilde{C}_{\mu\nu}\right]\right) \\ &= \Delta(\widetilde{C}_{\mu\nu}) \ , \end{aligned} \qquad (3.16)$$

since $[Q_{i\alpha},\widetilde{C}_{\mu\nu}] = 0$ by construction.

3.3 Representation on the algebra of superfunctions

Given a representation of the Hopf algebra $\mathcal{U}(\mathfrak{g})$ in an associative algebra consistent with the coproduct Δ, one needs to adjust the multiplication law after introducing a Drinfeld twist. If $\mathcal{F}^{-1}$ is the inverse of the element $\mathcal{F}\in\mathcal{U}(\mathfrak{g})\otimes\mathcal{U}(\mathfrak{g})$ generating the twist, the new product compatible with $\Delta^{\mathcal{F}}$ reads

$$a\star b \ := \ m^{\mathcal{F}}(a\otimes b) \ := \ m\circ\mathcal{F}^{-1}(a\otimes b) \ , \qquad (3.17)$$

where m denotes the ordinary product $m(a\otimes b) = ab$.

Let us now turn to the representation of the Hopf superalgebra $\mathcal{U}(\mathfrak{g})$ on the algebra $\mathcal{S} := C^\infty(\mathbb{R}^4)\otimes\Lambda_{4\mathcal{N}}$ of superfunctions on $\mathbb{R}^{4|4\mathcal{N}}$. On $\mathcal{S}$, we have the standard representation of the super Poincaré algebra in chiral coordinates $(y^\mu,\theta^{i\alpha},\bar{\theta}^{\dot{\alpha}}_i)$:

$$\begin{aligned} P_\mu f &= \mathrm{i}\partial_\mu f \ , \qquad M_{\mu\nu}f = \mathrm{i}(y_\mu\partial_\nu - y_\nu\partial_\mu)f \ , \\ Q_{i\alpha}f &= \frac{\partial}{\partial\theta^{i\alpha}}f \ , \qquad \bar{Q}^i_{\dot{\alpha}}f = \left(-\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}_i}f + 2\mathrm{i}\theta^{i\alpha}\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\right)f \ , \end{aligned} \qquad (3.18)$$

where f is an element of $\mathcal{S}$. After the twist, the multiplication m becomes the twist-adapted multiplication $m^{\mathcal{F}}$ (3.17), which reproduces the coordinate algebra of $\mathbb{R}^{4|4\mathcal{N}}_{\hbar}$, e.g. we have

$$\begin{aligned}\{\theta^{i\alpha},\theta^{j\beta}\}_\star :&= m^{\mathcal{F}}(\theta^{i\alpha}\otimes\theta^{j\beta}) + m^{\mathcal{F}}(\theta^{j\beta}\otimes\theta^{i\alpha}) \\ &= \theta^{i\alpha}\theta^{j\beta} + \frac{\hbar}{2}C^{i\alpha,j\beta} + \theta^{j\beta}\theta^{i\alpha} + \frac{\hbar}{2}C^{j\beta,i\alpha} \\ &= \hbar C^{i\alpha,j\beta}\ .\end{aligned} \tag{3.19}$$

Thus, we have constructed a representation of the euclidean super Poincaré algebra on $\mathbb{R}^{4|4\mathcal{N}}_{\hbar}$ by employing $\mathcal{S}_\star$, thereby making twisted supersymmetry manifest.

4. Applications

We saw in the above construction of the twisted euclidean super Poincaré algebra that our description is equivalent to the standard treatment of Moyal-Weyl-deformed superspace. We can therefore use it to define field theories via their lagrangians, substituting all products by star products, which then will be invariant under twisted super Poincaré transformations. This can be directly carried over to quantum field theories, replacing the products between operators by star products. Therefore, twisted super Poincaré invariance, in particular twisted supersymmetry, will always be manifest.

As a consistency check, we want to show that the tensor $C^{i\alpha,j\beta} := \{\theta^{i\alpha},\theta^{j\beta}\}_\star$ is invariant under twisted super Poincaré transformations before tackling more advanced issues. Furthermore, we want to relate the representation content of the deformed theory with that of the undeformed one by scrutinizing the Casimir operators of this superalgebra. Eventually, we will turn to supersymmetric Ward-Takahashi identities and their consequences for renormalizability.

4.1 Invariance of $C^{i\alpha,j\beta}$

The action of the twisted supersymmetry charge on $C^{i\alpha,j\beta}$ is given by

$$\begin{aligned}\hbar Q^{\mathcal{F}}_{k\gamma}C^{i\alpha,j\beta} &= Q^{\mathcal{F}}_{k\gamma}\left(\{\theta^{i\alpha},\theta^{j\beta}\}_\star\right) \\ :&= m^{\mathcal{F}}\circ\left(\Delta^{\mathcal{F}}(Q_{k\gamma})(\theta^{i\alpha}\otimes\theta^{j\beta}+\theta^{j\beta}\otimes\theta^{i\alpha})\right) \\ &= m^{\mathcal{F}}\circ\left(\Delta(Q_{k\gamma})(\theta^{i\alpha}\otimes\theta^{j\beta}+\theta^{j\beta}\otimes\theta^{i\alpha})\right) \\ &= m\circ\mathcal{F}^{-1}(\delta^i_k\delta^\alpha_\gamma\otimes\theta^{j\beta}+\delta^j_k\delta^\beta_\gamma\otimes\theta^{i\alpha}-\theta^{i\alpha}\otimes\delta^j_k\delta^\beta_\gamma-\theta^{j\beta}\otimes\delta^i_k\delta^\alpha_\gamma) \\ &= m(\delta^i_k\delta^\alpha_\gamma\otimes\theta^{j\beta}+\delta^j_k\delta^\beta_\gamma\otimes\theta^{i\alpha}-\theta^{i\alpha}\otimes\delta^j_k\delta^\beta_\gamma-\theta^{j\beta}\otimes\delta^i_k\delta^\alpha_\gamma) \\ &= 0\ .\end{aligned} \tag{4.1}$$

Similarly, we have

$$\begin{aligned}\hbar(\bar{Q}^k_{\dot\gamma})^{\mathcal{F}}C^{i\alpha,j\beta} &= m^{\mathcal{F}}\circ\left(\Delta^{\mathcal{F}}(\bar{Q}^k_{\dot\gamma})(\theta^{i\alpha}\otimes\theta^{j\beta}+\theta^{j\beta}\otimes\theta^{i\alpha})\right) \\ &= m^{\mathcal{F}}\circ\left(\Delta(\bar{Q}^k_{\dot\gamma})(\theta^{i\alpha}\otimes\theta^{j\beta}+\theta^{j\beta}\otimes\theta^{i\alpha})\right) \\ &= 0\ ,\end{aligned} \tag{4.2}$$

and

$$\hbar P^{\mathcal{F}}_{\mu\nu} C^{i\alpha,j\beta} = m^{\mathcal{F}} \circ \Big(\Delta(P_\mu)(\theta^{i\alpha} \otimes \theta^{j\beta} + \theta^{j\beta} \otimes \theta^{i\alpha})\Big) = 0 \ . \tag{4.3}$$

For the action of the twisted rotations and boosts, we get

$$\begin{aligned} \hbar M^{\mathcal{F}}_{\mu\nu} C^{i\alpha,j\beta} &= m^{\mathcal{F}} \circ \Big(\Delta^{\mathcal{F}}(M_{\mu\nu})(\theta^{i\alpha} \otimes \theta^{j\beta} + \theta^{j\beta} \otimes \theta^{i\alpha})\Big) \\ &= m \circ \mathcal{F}^{-1} \mathcal{F} \Delta(M_{\mu\nu}) \mathcal{F}^{-1} (\theta^{i\alpha} \otimes \theta^{j\beta} + \theta^{j\beta} \otimes \theta^{i\alpha}) \\ &= m(\mathbb{1} \otimes M_{\mu\nu} + M_{\mu\nu} \otimes \mathbb{1}) \Big((\theta^{i\alpha} \otimes \theta^{j\beta} + \theta^{j\beta} \otimes \theta^{i\alpha}) - \hbar C^{i\alpha,j\beta} \mathbb{1} \otimes \mathbb{1}\Big) \\ &= 0 \ , \end{aligned} \tag{4.4}$$

where we made use of $M_{\mu\nu} = \mathrm{i}(y_\mu \partial_\nu - y_\nu \partial_\mu)$. Thus, $C^{i\alpha,j\beta}$ is invariant under the twisted euclidean super Poincaré transformations, which is a crucial check of the validity of our construction.

4.2 Representation content

An important feature of noncommutative field theories was demonstrated recently [3, 4]: they share the same representation content as their commutative counterparts. Of course, one would expect this to also hold for non-anticommutative deformations, in particular since the superfields defined, e.g., in [11] on a deformed superspace have the same set of components as the undeformed ones.

To decide whether the representation content in our case is the same as in the commutative theory necessitates checking whether the twisted action of the Casimir operators $P^2 = P_\mu \star P^\mu$ and $\widetilde{C}^2 = \widetilde{C}_{\mu\nu} \star \widetilde{C}^{\mu\nu}$ on elements of $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ is altered with respect to the untwisted case. But since we have already shown in (3.10) and (3.16) that the actions of the operators P_μ and $\widetilde{C}_{\mu\nu}$ remain unaffected by the twist, it follows immediately that the operators P^2 and $\widetilde{C}^2$ are still Casimir operators in the twisted case. Together with the fact that the representation space considered as a module is not changed, this proves that the representation content is indeed the same.

4.3 Chiral rings and correlation functions

The chiral rings of operators in supersymmetric quantum field theories are cohomology rings of the supercharges $Q_{i\alpha}$ and $\bar{Q}^i_{\dot\alpha}$. Correlation functions which are built out of elements of a single such chiral ring have peculiar properties.

In [11], the anti-chiral ring was defined and discussed for non-anticommutative field theories. The chiral ring, however, lost its meaning: the supersymmetries generated by $\bar{Q}^i_{\dot\alpha}$ are broken, cf. (2.10), and therefore the vacuum is expected to be no longer invariant under this generator. Thus, the $\bar{Q}$-cohomology is not relevant for correlation functions of chiral operators.

In our approach to non-anticommutative field theory, *twisted* supersymmetry is manifest and therefore the chiral ring can be treated similarly to the untwisted case as we want to discuss in the following.

Let us assume that the Hilbert space $\mathcal{H}$ of our quantum field theory carries a representation of the euclidean super Poincaré algebra $\mathfrak{g}$, and that there is a unique, $\mathfrak{g}$ invariant vacuum state $|0\rangle$. Although the operators $Q_{i\alpha}$ and $\bar{Q}^i_{\dot{\alpha}}$ are not related via hermitean conjugation when considering supersymmetry on euclidean spacetime, it is still natural to assume that the vacuum is annihilated by both supercharges. The reasoning for this is basically the same as the one employed in [11] to justify the use of Minkowski superfields on euclidean spacetime: one can obtain a complexified supersymmetry algebra on euclidean space from a complexified supersymmetry algebra on Minkowski space.[10] Furthermore, it has been shown that in the non-anticommutative situation, just as in the ordinary undeformed case, the vacuum energy of the Wess-Zumino model is not renormalized [21].

We can now define the ring of chiral and anti-chiral operators by the relations

$$\{\!\![\bar{Q}, \mathcal{O}]\!\!\}_\star = 0 \qquad \text{and} \qquad \{\!\![Q, \bar{\mathcal{O}}]\!\!\}_\star = 0 \ , \tag{4.5}$$

respectively. In a correlation function built from chiral operators, $\bar{Q}$-exact terms, i.e. terms of the form $\{\!\![\bar{Q}, A]\!\!\}_\star$, do not contribute as is easily seen from

$$\begin{aligned}\langle \{\!\![\bar{Q}, A]\!\!\}_\star \star \mathcal{O}_1 \star \ldots \star \mathcal{O}_n\rangle =& \langle \{\!\![\bar{Q}, A \star \mathcal{O}_1 \star \ldots \star \mathcal{O}_n]\!\!\}_\star\rangle \pm \langle A \star \{\!\![\bar{Q}, \mathcal{O}_1]\!\!\}_\star \star \ldots \star \mathcal{O}_n\rangle \\ &\pm \ldots \pm \langle A \star \mathcal{O}_1 \star \ldots \star \{\!\![\bar{Q}, \mathcal{O}_n]\!\!\}_\star\rangle \\ =& \langle \bar{Q} A \star \mathcal{O}_1 \star \ldots \star \mathcal{O}_n\rangle \pm \langle A \star \mathcal{O}_1 \star \ldots \star \mathcal{O}_n \star \bar{Q}\rangle = 0 \ ,\end{aligned} \tag{4.6}$$

where we used that $\bar{Q}$ annihilates both $\langle 0|$ and $|0\rangle$. Therefore, the relevant operators in the chiral ring consist of the $\bar{Q}$-closed modulo the $\bar{Q}$-exact operators. The same argument holds for the anti-chiral ring after replacing $\bar{Q}$ with Q, namely

$$\begin{aligned}\langle \{\!\![Q, A]\!\!\}_\star \star \bar{\mathcal{O}}_1 \star \ldots \star \bar{\mathcal{O}}_n\rangle =& \langle \{\!\![Q, A \star \bar{\mathcal{O}}_1 \star \ldots \star \bar{\mathcal{O}}_n]\!\!\}_\star\rangle \pm \langle A \star \{\!\![Q, \bar{\mathcal{O}}_1]\!\!\}_\star \star \ldots \star \bar{\mathcal{O}}_n\rangle \\ &\pm \ldots \pm \langle A \star \bar{\mathcal{O}}_1 \star \ldots \star \{\!\![Q, \bar{\mathcal{O}}_n]\!\!\}_\star\rangle \\ =& \langle Q A \star \bar{\mathcal{O}}_1 \star \ldots \star \bar{\mathcal{O}}_n\rangle \pm \langle A \star \bar{\mathcal{O}}_1 \star \ldots \star \bar{\mathcal{O}}_n \star Q\rangle = 0 \ .\end{aligned} \tag{4.7}$$

4.4 Twisted supersymmetric Ward-Takahashi identities

The above considered properties of correlation functions are particularly useful since they imply a twisted supersymmetric Ward-Takahashi identity: any derivative with respect to the bosonic coordinates of an anti-chiral operator annihilates a purely chiral or anti-chiral correlation function. This is due to the fact that $\partial \sim \{Q, \bar{Q}\}$ and therefore any derivative gives rise to a Q-exact term, which causes an anti-chiral correlation function to vanish. Analogously, the bosonic derivatives of chiral correlation functions vanish. Thus, the correlation functions are independent of the bosonic coordinates, and we can move the operators to a far distance of each other. This causes the correlation function to factorize[11]:

$$\langle \bar{\mathcal{O}}_1(x_1) \star \ldots \star \bar{\mathcal{O}}_n(x_n)\rangle = \langle \bar{\mathcal{O}}_1(x_1^\infty)\rangle \star \ldots \star \langle \bar{\mathcal{O}}_n(x_n^\infty)\rangle \ . \tag{4.8}$$

and such a correlation function therefore does not contain any contact terms. This phenomenon is called *clustering* in the literature.

[10] One can then perform all superspace calculations and impose suitable reality conditions on the component fields in the end.

[11] This observation has first been made in [22].

Another direct consequence of (4.6) is the holomorphic dependence of the chiral correlation functions on the coupling constants, i.e.

$$\frac{\partial}{\partial\bar{\lambda}}\langle\mathcal{O}_1\star\ldots\star\mathcal{O}_n\rangle = 0\ . \tag{4.9}$$

As an illustrative example for this, consider the case of a $\mathcal{N}=1$ superpotential 'interaction'

$$\mathcal{L}_{\mathrm{W}} = \int \mathrm{d}^2\theta\lambda\Phi + \int \mathrm{d}^2\bar{\theta}\bar{\lambda}\bar{\Phi}\ , \tag{4.10}$$

where $\Phi = \phi(y) + \sqrt{2}\theta^\alpha\psi_\alpha(y) + \theta^2 F(y)$ is a chiral superfield and one of the supersymmetry transformations is given by $\{Q_\alpha, \psi_\beta\} = \varepsilon_{\alpha\beta}F$. Then we have

$$\begin{aligned}\frac{\partial}{\partial\bar{\lambda}}\langle\mathcal{O}_1\star\ldots\star\mathcal{O}_n\rangle = \int \mathrm{d}^4y\mathrm{d}^2\bar{\theta}\langle\mathcal{O}_1\star\ldots\star\mathcal{O}_n\star\bar{\Phi}\rangle \ &= \int \mathrm{d}^4y\langle\mathcal{O}_1\star\ldots\star\mathcal{O}_n F\rangle \\ = \int \mathrm{d}^4y\langle\mathcal{O}_1\star\ldots\star\mathcal{O}_n\{\bar{Q}_{\dot{\alpha}},\bar{\psi}^{\dot{\alpha}}\}\rangle \ &= \ 0\ .\end{aligned} \tag{4.11}$$

4.5 Comments on non-renormalization theorems

A standard perturbative non-renormalization theorem[12] for $\mathcal{N}=1$ supersymmetric field theory states that every term in the effective action can be written as an integral over $\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}$. It has been shown in [21] that this theorem also holds in the non-anticommutative case. The same is then obviously true in our case of twisted and therefore unbroken supersymmetry, and the proof carries through exactly as in the ordinary case.

Furthermore, in a supersymmetric non-linear sigma-model, the superpotential is not renormalized. A nice argument for this fact was given in [24]. Instead of utilizing Feynman diagrams and supergraph techniques, one makes certain naturalness assumptions about the effective superpotential. These assumptions turn out to be strong enough to enforce a non-perturbative non-renormalization theorem.

In the following, let us demonstrate this argument in a simple case, following closely [25]. Take a non-linear sigma model with superpotential

$$\mathcal{W} = \frac{1}{2}m\Phi^2 + \frac{1}{3}\lambda\Phi^3\ , \tag{4.12}$$

where $\Phi = \phi + \sqrt{2}\theta\psi + \theta\theta F$ is an ordinary chiral superfield. The assumptions we impose on the effective action are the following:

- ▷ Supersymmetry is also a symmetry of the effective superpotential.
- ▷ The effective superpotential is holomorphic in the coupling constants.
- ▷ Physics is smooth and regular under the possible weak-coupling limits.
- ▷ The effective superpotential preserves the $\mathsf{U}(1)\times\mathsf{U}(1)_R$ symmetry of the original superpotential with charge assignments $\Phi : (1,1)$, $m : (-2,0)$, $\lambda : (-3,-1)$ and $\mathrm{d}^2\theta : (0,-2)$.

[12] For more details and a summary of non-renormalization theorems, see [23].

It follows that the effective superpotential must be of the form

$$\mathcal{W}_{\text{eff}} = m\Phi W\left(\frac{\lambda\Phi}{m}\right) = \sum_i a_i \lambda^i m^{1-i} \Phi^{i+2} \ , \tag{4.13}$$

where W is an arbitrary holomorphic function of its argument. Regularity of physics in the two weak-coupling limits $\lambda \to 0$ and $m \to 0$ then implies that $\mathcal{W}_{\text{eff}} = \mathcal{W}$.

To obtain an analogous non-renormalization theorem in the non-anticommutative setting, we make similar assumptions about the effective superpotential as above. We start from

$$\mathcal{W}_\star = \frac{1}{2} m\Phi \star \Phi + \frac{1}{3}\lambda\Phi \star \Phi \star \Phi \ , \tag{4.14}$$

and assume the following:

- ▷ *Twisted* supersymmetry is a symmetry of the effective superpotential. Note that this assumption is new compared to the discussion in [21]. Furthermore, arguments substantiating that the effective action can always be written in terms of star products have been given in [26].
- ▷ The effective superpotential is holomorphic in the coupling constants. (This assumption is equally natural as in the supersymmetric case, since it essentially relies on the existence of chiral and anti-chiral rings, which we proved above for our setting.)
- ▷ Physics is smooth and regular under the possible weak-coupling limits.
- ▷ The effective superpotential preserves the $\mathsf{U}(1) \times \mathsf{U}(1)_R$ symmetry of the original superpotential with charge assignments $\Phi : (1,1)$, $m : (-2,0)$, $\lambda : (-3,-1)$, $\mathrm{d}^2\theta$: $(0,-2)$ and, additionally, $C^{i\alpha,j\beta} : (0,2)$, $|C| \sim C^{i\alpha,j\beta} C_{i\alpha,j\beta} : (0,4)$.

At first glance, it seems that one can now construct more $\mathsf{U}(1) \times \mathsf{U}(1)_R$-symmetric terms in the effective superpotential due to the new coupling constant C; however, this is not true. Taking the $C \to 0$ limit, one immediately realizes that C can never appear in the denominator of any term. Furthermore, it is not possible to construct a term containing C in the nominator, which does not violate the regularity condition in at least one of the other weak-coupling limits. Altogether, we arrive at an expression similar to (4.13)

$$\mathcal{W}_{\text{eff},\star} = \sum_i a_i \lambda^i m^{1-i} \Phi^{\star i+2} \ , \tag{4.15}$$

and find that $\mathcal{W}_{\text{eff},\star} = \mathcal{W}_\star$.

To compare this result with the literature, first note that, in a number of papers, it has been shown that quantum field theories in four dimensions with $\mathcal{N} = \frac{1}{2}$ supersymmetry are renormalizable to all orders in perturbation theory [27 − 32]. This even remains true for generic $\mathcal{N} = \frac{1}{2}$ gauge theories with arbitrary coefficients, which do not arise as a $\star$-deformation of $\mathcal{N} = 1$ theories. However, the authors of [21, 28], considering the non-anticommutative Wess-Zumino model we discussed above, add certain terms to the action by hand, which seem to be necessary for the model to be renormalizable. This would clearly contradict our result $\mathcal{W}_{\text{eff},\star} = \mathcal{W}_\star$. We conjecture, that this contradiction is merely a seeming one and that it is resolved by a resummation of all the terms in the perturbative

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expansion. A similar situation was encountered in [26], where it was found that one could not write certain terms of the effective superpotential using star products, as long as they were considered separately. This obstruction, however, vanished after a resummation of the complete perturbative expansion and the star product was found to be sufficient to write down the complete effective superpotential.

Clearly, the above result is stricter than the result obtained in [21], where less constraint terms in the effective superpotential were assumed. However, we should stress that it is still unclear to what extend the above assumptions on $\mathcal{W}_{\text{eff},\star}$ are really natural. This question certainly deserves further and deeper study, which we prefer to leave to future work.

5. Conclusions and outlook

We constructed a Drinfeld-twisted Hopf superalgebra and used this setup to study certain aspects of $\mathcal{N} = 1/2$ supersymmetric quantum field theories and their $\mathcal{N}$-extended variants. In particular, we scrutinized the consequences of this twisting, i.e. the introduction of a twisted (euclidean) super Poincaré symmetry, on various important structures of supersymmetric QFTs, such as the cohomology ring of chiral operators and related Ward-Takahashi identities. We found that in our framework, a twisted version of these notions can be retrieved and can thus be used to simplify calculations.

Furthermore, we discussed a number of 'naturalness' assumptions on the deformed superpotential which can lead to non-perturbative non-renormalization theorems similar to those in the $\mathcal{N} = 1$ supersymmetric case. Granted these assumptions, these theorems bring about many potential simplifications in higher loop calculations within $\mathcal{N} = 1/2$ supersymmetric QFT. More work is needed, however, to clarify the situation here.

Possible future studies might include Drinfeld-twisted superconformal invariance. Studying twist-deformed superconformal field theories, following the discussion of twisted conformal invariance in [8], could potentially yield further interesting results. Moreover, the ideas presented above may prove valuable for introducing a non-anticommutative deformation of supergravity. Building upon the discussion presented in [10], one could try to construct a local version of the twisted supersymmetry. The latter proposal appears interesting to us and certainly deserves further investigation.

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Note added

After finishing this paper, the article [33] appeared, in which an analogous construction was discussed for $\mathcal{N} = (1,1)$ supersymmetry.

A. Definitions

Recall that a *Hopf algebra* is an algebra H over a field k together with a *product* m, a *unit* $\mathbb{1}$, a *coproduct* $\Delta : H \to H \otimes H$ satisfying $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$, a *counit* $\varepsilon : H \to k$ satisfying $(\varepsilon \otimes \mathrm{id})\Delta = \mathrm{id}$ and $(\mathrm{id} \otimes \varepsilon)\Delta = \mathrm{id}$ and an *antipode* $S : H \to H$ satisfying $m(S \otimes \mathrm{id})\Delta = \varepsilon\mathbb{1}$ and $m(\mathrm{id} \otimes S)\Delta = \varepsilon\mathbb{1}$. The maps Δ, ε and S are unital maps, that is $\Delta(\mathbb{1}) = \mathbb{1} \otimes \mathbb{1}$, $\varepsilon(\mathbb{1}) = 1$ and $S(\mathbb{1}) = \mathbb{1}$.

A *supervector space* is a $\mathbb{Z}_2$-graded vector space, i.e., one can decompose a supervector space into the direct sum of an even and an odd subspace. If an element v of a supervector space is contained in the even or the odd subspace, we write $\tilde{v} = 0$ or $\tilde{v} = 1$, respectively. A *superalgebra* is a supervector space endowed with an associative multiplication respecting the grading (i.e. $\widetilde{ab} \equiv \tilde{a} + \tilde{b} \mod 2$) and a unit $\mathbb{1}$ with $\tilde{\mathbb{1}} = 0$. On superalgebras, we define the *graded commutator* by $[\![a, b]\!] := ab - (-1)^{\tilde{a}\tilde{b}}ba$.

We fix the following rule for the interplay between the multiplication and the tensor product $\otimes$ in a superalgebra:

$$(a_1 \otimes a_2)(b_1 \otimes b_2) = (-1)^{\tilde{a}_2\tilde{b}_1}(a_1b_1 \otimes a_2b_2) \ . \tag{A.1}$$

A superalgebra is called a *Hopf superalgebra* if it is endowed with a graded coproduct[13] Δ and a counit ε, both of which are graded algebra morphisms, i.e.

$$\Delta(ab) = \sum(-1)^{\tilde{a}_{(2)}\tilde{b}_{(1)}} a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)} \qquad \text{and} \qquad \varepsilon(ab) = \varepsilon(a)\varepsilon(b) \ , \tag{A.2}$$

and an antipode S which is a graded algebra anti-morphism, i.e.

$$S(ab) = (-1)^{\tilde{a}\tilde{b}}S(b)S(a) \ . \tag{A.3}$$

As usual, one furthermore demands that Δ, ε and S are unital maps, that Δ is coassociative and that ε and S are counital. For more details, see [34] and references therein.

B. Extended graded Baker-Campbell-Hausdorff formula

First, note that $\mathrm{e}^{A\otimes B}\mathrm{e}^{-A\otimes B}$ is indeed equal to $\mathbb{1} \otimes \mathbb{1}$ for any two elements A, B of a superalgebra. This is clear for $\tilde{A} = 0$ or $\tilde{B} = 0$. For $\tilde{A} = \tilde{B} = 1$ it is most instructively gleaned from

$$\left(\mathbb{1} \otimes \mathbb{1} + A \otimes B - \frac{1}{2}A^2 \otimes B^2 + \dots\right)\left(\mathbb{1} \otimes \mathbb{1} - A \otimes B - \frac{1}{2}A^2 \otimes B^2 - \dots\right) = \mathbb{1}\otimes\mathbb{1} \ . \tag{B.1}$$

[13] In Sweedler's notation with $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$, this amounts to $\tilde{a} \equiv \tilde{a}_{(1)} + \tilde{a}_{(2)} \mod 2$.

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Now, for elements A_I, B_J, D of a graded algebra, where the parities of the elements A_I and B_J are all equal $\tilde{A} = \tilde{A}_I = \tilde{B}_J$ and $\{\!\!\{A_I, A_J\}\!\!\} = \{\!\!\{B_I, B_J\}\!\!\} = 0$, we have the relation

$$\begin{aligned} &\mathrm{e}^{C^{IJ}A_I\otimes B_J}\,(D\otimes\mathbb{1})\,\mathrm{e}^{-C^{KL}A_K\otimes B_L} = \\ &\quad= \sum_{n=0}^{\infty}\frac{(-1)^{n\tilde{A}\tilde{D}+\frac{n(n-1)}{2}\tilde{A}}}{n!}C^{I_1J_1}\dots C^{I_nJ_n}\{\!\!\{A_{I_1},\{\!\!\{\dots\{\!\!\{A_{I_n},D\}\!\!\}\}\!\!\}\}\!\!\}\otimes B_{J_1}\dots B_{J_n}\ . \end{aligned} \tag{B.2}$$

Proof: To verify this relation, one can simply adapt the well-known iterative proof via a differential equation. First note that

$$\mathrm{e}^{\lambda C^{IJ}A_I\otimes B_J}(C^{KL}A_K\otimes B_L) = (C^{KL}A_K\otimes B_L)\mathrm{e}^{\lambda C^{IJ}A_I\otimes B_J}\ . \tag{B.3}$$

Then define the function

$$F(\lambda)\ :=\ \mathrm{e}^{\lambda C^{IJ}A_I\otimes B_J}(D\otimes 1)\mathrm{e}^{-\lambda C^{KL}A_K\otimes B_L}\ , \tag{B.4}$$

which has the derivative

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\lambda}F(\lambda) =&(C^{MN}A_M\otimes B_N)\mathrm{e}^{\lambda C^{IJ}A_I\otimes B_J}(D\otimes 1)\mathrm{e}^{-\lambda C^{KL}A_K\otimes B_L}- \\ &-\mathrm{e}^{\lambda C^{IJ}A_I\otimes B_J}(D\otimes 1)\mathrm{e}^{-\lambda C^{KL}A_K\otimes B_L}(C^{MN}A_M\otimes B_N)\ . \end{aligned} \tag{B.5}$$

Thus, we have the identity $\frac{\mathrm{d}}{\mathrm{d}\lambda}F(\lambda) = [(C^{MN}A_M\otimes B_N), F(\lambda)]$, which, when applied recursively together with the Taylor formula, leads to

$$F(1) = \sum_{n=0}^{\infty}\frac{1}{n!}\left[C^{I_1J_1}A_{I_1}\otimes B_{J_1}\left[\dots\left[C^{I_nJ_n}A_{I_n}\otimes B_{J_n}, D\otimes\mathbb{1}\right]\dots\right]\right]\ . \tag{B.6}$$

Also recursively, one easily checks that

$$\begin{aligned} &\left[C^{I_1J_1}A_{I_1}\otimes B_{J_1}\left[\dots\left[C^{I_nJ_n}A_{I_n}\otimes B_{J_n}, D\otimes\mathbb{1}\right]\dots\right]\right] = \\ &\qquad= (-1)^{\tilde{A}\tilde{D}}(-1)^{\kappa}C^{I_1J_1}\dots C^{I_nJ_n}\{\!\!\{A_{I_1},\{\!\!\{\dots\{\!\!\{A_{I_n},D\}\!\!\}\}\!\!\}\}\!\!\}\otimes B_{J_1}\dots B_{J_n}\ , \end{aligned} \tag{B.7}$$

where κ is given by $\kappa = (n-1)\tilde{A} + (n-2)\tilde{A} + \ldots + \tilde{A}$. Furthermore, we have

$$(-1)^{\kappa} = (-1)^{n^2-\sum_{i=1}^{n}i} = (-1)^{n^2+\sum_{i=1}^{n}i} = (-1)^{\frac{n(n-1)}{2}}\ , \tag{B.8}$$

which, together with the results above, proves formula (B.2). This extended graded Baker-Campbell-Hausdorff formula also generalizes straightforwardly to the case when $D\otimes\mathbb{1}$ is replaced by $\mathbb{1}\otimes D$.

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